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COMPRESSION OF AN ELASTIC ROLLER BETWEEN TWO RIGID PLATES

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Summary.

A closed solution — exact within two-dimensional linear elastostatics — is deduced for the problem appropriate to the compression of an elastic circular cylinder between two smooth, flat and parallel, rigid plates. The boundary displacements obtained for the cylinder involve elliptic integrals, whereas its stress field is given in terms of elementary functions exclusively. The results found for the distribution of the contact pressure and for the width of the contact zone are compared with the corresponding predictions of Hertz's approximate theory, for which elementary corrections are determined by asymptotic means. Analogous corrections are established for a previously available approximate estimate of the diametral compression undergone by the roller, which remains indeterminate in the Hertz treatment of this two-dimensional contact problem.

Introduction.

As is well known and amply confirmed by experiment, the approximate treatment due to Hertz [1][1] of the frictionless contact between two elastic

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[1] See also Love [2], Chapter VIII.

solids of essentially arbitrary initial shape yields highly satisfactory results for the local deformations and stresses in the vicinity of the contact region. This is true, in particular, of Hertz's conclusions regarding the size and shape of the region of contact, as well as of the Hertzian contact-pressure distribution[1]. In addition, the Hertz theory leads to a very good approximation for the "normal approach" of sufficiently remote parts of the two contiguous bodies.

For the special case of contact between cylindrical bodies (plane contact problem), equally favorable local results are obtainable from Hertz's three-dimensional solution through a limit process[2] in which the major principal diameter of the contact ellipse is permitted to grow beyond bounds. On the other hand, the normal approach is left indeterminate by the Hertz theory of the plane contact problem. This deficiency is ultimately traceable to a familiar pathology associated with the two-dimensional (plane-strain or generalized plane-stress) solution for a half-plane under unbalanced edge tractions, the displacement field of which is logarithmically unbounded at infinity.

To eliminate the indeterminacy alluded to above, it is essential to relinquish a purely local treatment of the plane contact problem and take into account the entire shape and size of at least one of the bodies in contact. Of particular interest in this connection — because of its relevance to the design of roller bearings — is the problem occasioned by the

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[1]Although Hertz did not pursue the explicit determination of the local stress fields to which the ensuing pressure distribution gives rise in either body, these fields were subsequently explored by various investigators. See Timoshenko and Goodier [3], p. 376 for references.

[2]See, for example, [3], p. 381. Alternatively, the same predictions have been deduced directly by Föppl [4] and Poritsky [5] on the basis of the two-dimensional analogue of Hertz's approximative scheme.

compression of a circular roller of finite radius between two (not necessarily identical) cylindrical bodies, whose elastic properties may be distinct from those of the roller. Here the normal approach between the two outside bodies, i.e., the diametral compression of the roller, is an item of obvious physical concern.

Approximate calculations for this diametral compression, resting on the Hertzian results for the contact width and the contact pressure, have been carried out by various investigators. In this manner Föppl [6] (p. 324) was led to propose a frequently cited formula, based on the assumption that all three bodies are of the same material and that the contact surfaces of both outside bodies are initially plane. Much more recently, Johnson [7][1] deduced by conspicuously elementary means a similar estimate that is free from Föppl's restrictive assumptions.

A more ambitious effort pertaining to the plane contact problem under discussion is due to Loo [8], who dealt with the case in which both outside bodies are identical circular cylinders of arbitrary radius and not necessarily the same material as the roller. The primary purpose of [8] is a refinement of the Hertz theory that avoids the local geometric approximations of the initial body shapes underlying the Hertzian treatment of the problem considered. Unfortunately, the analysis in [8] falls short of this aim since the terms neglected in the integral equation to which the problem is eventually reduced are of the same order of magnitude as those neglected in the Hertz theory from the very start. Accordingly, the corrections of the Hertzian results found in [8] are spurious, as is borne

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[1] We are indebted to K. L. Johnson of Cambridge University for making available to us a section of his as yet unpublished monograph on contact mechanics.

out by the results obtained in the present investigation. Nevertheless, Loo arrives at a first approximation for the diametral compression of the roller that is consistent with our own findings.

This paper is devoted entirely to a limiting case of the problems considered by Johnson [7] and Loo [8]. Thus, we are concerned with the stresses and deformations in a homogeneous and isotropic elastic circular cylinder that is compressed between two parallel, flat and frictionless, rigid plates. A solution to this problem, exact within linear two-dimensional elastostatics, is established in closed form. Apart from the intrinsic interest of a rigorous closed global solution to a contact problem, the results supply a theoretical confirmation of the high degree of accuracy inherent in the corresponding Hertzian results, for which explicit elementary corrections are determined. Further, the diametral compression of the roller obtained here permits — for the limiting case under consideration — a reliable appraisal and improvement of the analogous approximate formulas mentioned earlier.

In Section 1 we formulate the problem to be solved and reduce the latter to a singular integral equation, with certain side conditions, for the unknown contact-pressure distribution. This integral equation is, in turn, reduced to a mixed Hilbert problem for the half-plane. In Section 2 we show that the known general solution of the subsidiary Hilbert problem may be adapted to yield the solution to the original problem. In Section 3 we give the final result for the contact pressure and arrive at closed elliptic-integral representations for the boundary displacements. In particular, this section includes the results for the contact width and the diametral compression of the roller. Here we also establish elementary asymptotic estimates for the physically most important quantities. These estimates afford a convenient comparison with previously available approximate results. Section 4 contains the determination of the complete stress distribution in the roller in closed elementary form. The paper concludes with an Appendix on the closely related problem of a roller under two opposing flat rigid punches, the solution of which is shown to be easily obtainable through an appropriate modification of the analysis carried out in the body of the paper.

1. Formulation of problem. Reduction to a mixed Hilbert problem for the half-plane.

We proceed now to the mathematical statement of the mixed boundary-value problem that constitutes our objective. Let R be the open circular cross-section of the roller prior to its deformation, call r_o the radius of the boundary ∂R, and choose rectangular cartesian coordinates (x_1, x_2) with the origin at the center of R, as indicated in Figure 1. Next, designate by μ and ν, in this order, the shear modulus and Poisson's ratio of the roller, denoting by $\underset{\sim}{u}(\underset{\sim}{x})$ and $\underset{\sim}{\sigma}(\underset{\sim}{x})$ the respective values of the displacement vector and of the stress tensor at a point of R with the position vector $\underset{\sim}{x}$. The foregoing fields of displacement and stress are induced by the compression of the roller between two rigid plates, the plane faces of which are taken to be perpendicular to the x_1-axis.

We require $\underset{\sim}{u}$ and $\underset{\sim}{\sigma}$ to be continuously differentiable on R and continuous on its closure $\overline{R}$, except possibly for singularities of $\underset{\sim}{\sigma}$ at the four boundary points characterized by

$$|x_1| = r_o \cos\theta_o \qquad (0 < \theta_o < \pi/2) \; , \tag{1.1}$$

which correspond to the endpoints of the two arcs of ∂R that are brought into complete contact with the plate faces as a result of the deformation of the roller[1] (Figure 1). Throughout R the displacements and stresses are to satisfy the classical two-dimensional field equations of plane strain for the given elastic constants, in the absence of body forces. Further, in order to simplify the remaining specification of the problem, we stipulate that the fields $\underset{\sim}{u}$ and $\underset{\sim}{\sigma}$ be symmetric with respect to both coordinate axes.

Let (r, θ) be the polar coordinates defined by

$$x_1 = r\cos\theta \quad , \quad x_2 = r\sin\theta \quad (0 \le r \le r_o \, , \; -\pi \le \theta < \pi) \tag{1.2}$$

and suppose that

$$s_i(\theta) = \sigma_{ij}(\underset{\sim}{x}(r_o, \theta)) n_j(\underset{\sim}{x}(r_o, \theta)) \quad (-\pi \le \theta < \pi) \; , \tag{1.3}$$ [2]

where $\underset{\sim}{n}$ is the unit outward normal vector of ∂R, so that $\underset{\sim}{s}$ is the surface-traction vector associated with $\underset{\sim}{\sigma}$ on ∂R. Also, for convenience, write

$$\underset{\sim}{v}(\theta) = \underset{\sim}{u}(\underset{\sim}{x}(r_o, \theta)) \quad (-\pi \le \theta < \pi) \tag{1.4}$$

for the boundary displacements. In view of the notation (1.3), (1.4) and by virtue of the symmetry requirement introduced earlier, it suffices to state the boundary conditions in the form

$$s_1(\theta) = 0 \quad (\theta_o < \theta \le \tfrac{\pi}{2}) \quad , \quad s_2(\theta) = 0 \quad (0 \le \theta \le \tfrac{\pi}{2}) \; , \tag{1.5}$$

$$v_1(\theta) = r_o(\cos\theta_* - \cos\theta) \quad (0 \le \theta \le \theta_o) \; . \tag{1.6}$$

Equations (1.5) assert that no loads are applied to the surface of the roller outside the contact zone and that the contact is frictionless. On the other hand, (1.6) represents the contact condition, in which $\theta_* \, (0 < \theta_* < \frac{\pi}{2})$ is an initially unknown parameter that governs the final position of the

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[1] Thus $2\theta_o$ is the central angle subtended by each of the two "contact arcs".

[2] Here the subscripts i, j have the range (1, 2) and the usual summation convention is employed. If $\underset{\sim}{a}$ is a vector, we write a_i for its cartesian components; the analogous notation is used for second-order tensors.

plate faces; indeed, as is clear from Figure 1,

$$\Delta = 2r_o(1-\cos\theta_*) . \tag{1.7}$$

provided Δ is the diametral compression of the roller. To the boundary conditions (1.5), (1.6) one needs to adjoin the <u>separation condition</u>

$$v_1(\theta) \le r_o(\cos\theta_* - \cos\theta) \quad (\theta_o \le \theta \le \frac{\pi}{2}) , \tag{1.8}$$

according to which the deformed roller does not penetrate the compressing rigid plates, together with the <u>condition of unilateral constraint</u>

$$s_1(\theta) \le 0 \quad (0 \le \theta < \theta_o) , \tag{1.9}$$

which precludes the transmission of tensile tractions across the contact zone. Finally, we require s_1 to be integrable, as well as continuously differentiable, on $[0, \theta_o)$ and impose the <u>load condition</u>

$$-2r_o \int_0^{\theta_o} s_1(\theta) d\theta = P , \tag{1.10}$$

where P is the given resultant scalar load per unit generator-length exerted by the plates upon the roller.

In summary, we are to find displacements and stresses $\underset{\sim}{u}$, $\underset{\sim}{\sigma}$ of the specified regularity and symmetry on $\overline{R}$, together with the values of the unknown parameters θ_o and θ_*, subject to the homogeneous field equations of plane strain on R accompanied by conditions (1.5), (1.6), (1.8), (1.9), and (1.10).

We note here that the foregoing formulation, though rigorous within linearized elastostatics, is different from the standard statement of boundary-value problems in this theory. In the conventional formulation of such problems the boundary conditions are referred wholly to the undeformed boundary, whereas the second of (1.5) in conjunction with (1.6) amount to the specification of the tractions tangential, and of the displacements normal, to a portion of the deformed surface of the roller. This departure from the usual practice is necessitated by the fact that in the particular physical circumstances under consideration the displacements normal to the original cylindrical boundary are not known beforehand, while those perpendicular to the plate faces are determinate up to an initially unknown constant whose value is ultimately deducible with the aid of the load condition (1.10).

With a view toward a convenient reduction of the problem at hand, let $\overset{*}{\underset{\sim}{u}}(\underset{\sim}{x}, \alpha)$ and $\overset{*}{\underset{\sim}{\sigma}}(\underset{\sim}{x}, \alpha)$, for $0 \le \alpha < \frac{\pi}{2}$ and every $\underset{\sim}{x}$ in $\overline{R}$ such that $|x_1| \neq r_o \cos\alpha$, denote the displacement and stress field of the plane-strain solution appropriate to the given roller under four compressive surface line-loads (Figure 2) of unit lineal density, applied parallel to the x_1-axis along the generators situated at

$$\left.\begin{aligned} &x_1 = r_o \cos\alpha \ , \quad x_2 = \pm r_o \sin\alpha \ , \\ &x_1 = -r_o \cos\alpha \ , \quad x_2 = \pm r_o \sin\alpha \ . \end{aligned}\right\} \quad (1.11)$$

We now set

$$p(\theta) = -s_1(\theta) \quad (0 \le \theta < \theta_o) \ , \qquad (1.12)$$

whence p is the desired contact pressure, and assume the solution to the contact problem under consideration in the form

$$\underset{\sim}{u}(\underset{\sim}{x}) = r_o \int_0^{\theta_o} \overset{*}{\underset{\sim}{u}}(\underset{\sim}{x}, \alpha) p(\alpha) d\alpha \ , \quad \underset{\sim}{\sigma}(\underset{\sim}{x}) = r_o \int_0^{\theta_o} \overset{*}{\underset{\sim}{\sigma}}(\underset{\sim}{x}, \alpha) p(\alpha) d\alpha \ . \qquad (1.13)$$

The auxiliary singular solution $\overset{*}{\underset{\sim}{u}}, \overset{*}{\underset{\sim}{\sigma}}$, which plays the role of a Green's function in the present context, is readily obtainable by means of an appropriate symmetrization of the well-known plane-strain solution, due to Hertz [9], for a circular cylinder under two equal, opposite and coplanar surface line-loads of constant density. Using Muskhelishvili's [10] (§ 80a) complex version of Hertz's solution one finds in this manner, upon putting

$$z = x_1 + ix_2 = r\exp(i\theta) \quad , \quad \mathfrak{z} = r_0 \exp(i\alpha) \tag{1.14}$$

and on writing $\bar{z}$, $\bar{\mathfrak{z}}$ for the complex conjugates of z, $\mathfrak{z}$:

$$\left.\begin{aligned} &\mathring{\sigma}_{11}(\underset{\sim}{x},\alpha)+\mathring{\sigma}_{22}(\underset{\sim}{x},\alpha)=4\mathrm{Re}\{\mathring{\Phi}(z,\mathfrak{z})\} \ , \\ &\mathring{\sigma}_{22}(\underset{\sim}{x},\alpha)-\mathring{\sigma}_{11}(\underset{\sim}{x},\alpha)+2i\mathring{\sigma}_{12}(\underset{\sim}{x},\alpha)=2[\bar{z}\mathring{\Phi}'(z,\mathfrak{z})+\mathring{\Psi}(z,\mathfrak{z})] \ , \end{aligned}\right\} \tag{1.15}$$

if $\mathring{\Phi}'$ is the derivative of $\mathring{\Phi}$ with respect to z and

$$\left.\begin{aligned} &\mathring{\Phi}(z,\mathfrak{z})=\frac{1}{2\pi}\left[\frac{1}{z-\mathfrak{z}}+\frac{1}{z-\bar{\mathfrak{z}}}-\frac{1}{z+\mathfrak{z}}-\frac{1}{z+\bar{\mathfrak{z}}}-\frac{z+\bar{\mathfrak{z}}}{r_0^2}\right] \ , \\ &\mathring{\Psi}(z,\mathfrak{z})=-\frac{1}{2\pi}\left[\frac{1}{z-\mathfrak{z}}+\frac{1}{z-\bar{\mathfrak{z}}}-\frac{1}{z+\mathfrak{z}}-\frac{1}{z+\bar{\mathfrak{z}}}-\frac{\bar{\mathfrak{z}}}{(z-\mathfrak{z})^2}-\frac{\mathfrak{z}}{(z-\bar{\mathfrak{z}})^2}-\frac{\bar{\mathfrak{z}}}{(z+\mathfrak{z})^2}-\frac{\mathfrak{z}}{(z+\bar{\mathfrak{z}})^2}\right] \ ; \end{aligned}\right\} \tag{1.16}$$

$$\left.\begin{aligned} &\mathring{u}_1(\underset{\sim}{x},\alpha)+i\mathring{u}_2(\underset{\sim}{x},\alpha)=-\frac{1}{4\pi\mu}\left[(3-4\nu)\log\frac{(z+\mathfrak{z})(z+\bar{\mathfrak{z}})}{(z-\mathfrak{z})(z-\bar{\mathfrak{z}})}\right. \\ &\left.+\log\frac{(\bar{z}+\mathfrak{z})(\bar{z}+\bar{\mathfrak{z}})}{(\bar{z}-\mathfrak{z})(\bar{z}-\bar{\mathfrak{z}})}+\frac{z-\mathfrak{z}}{\bar{z}-\bar{\mathfrak{z}}}+\frac{z-\bar{\mathfrak{z}}}{\bar{z}-\mathfrak{z}}-\frac{z+\mathfrak{z}}{\bar{z}+\bar{\mathfrak{z}}}-\frac{z+\bar{\mathfrak{z}}}{\bar{z}+\mathfrak{z}}-\frac{4(1-2\nu)}{r_0}z\cos\alpha\right] . \end{aligned}\right\} \tag{1.17}$$

The first logarithm in (1.17) is to be interpreted as that branch which vanishes at $z=0$ and is analytic in the entire z-plane cut along the four horizontal rays joining the points $\mathfrak{z}$, $\bar{\mathfrak{z}}$, $-\mathfrak{z}$, $-\bar{\mathfrak{z}}$ to the point at infinity (see Figure 2); the second logarithm is the complex conjugate of the first[1].

It is not difficult to show that the fields defined by (1.13) together with (1.14) to (1.17) possess the requisite smoothness and symmetry on $\bar{R}$ and satisfy the field equations of plane strain on R, as well as the boundary conditions (1.5), for every p that is integrable and continuously

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[1] This normalization assures that $\underset{\sim}{\mathring{u}}$ and the scalar rotation vanish at the origin.

differentiable on $[0, \theta_o)$. We seek next to determine p in such a way that conditions (1.6), (1.8), (1.9), (1.10) also hold true.

With this objective in mind we adopt the notation

$$\underset{\sim}{\vartheta}(\theta, \alpha)=\underset{\sim}{\mathring{u}}(\underset{\sim}{x}(r_o, \theta), \alpha) \qquad (0\leq\theta\leq\frac{\pi}{2}\ ,\ 0\leq\alpha<\frac{\pi}{2}\ ,\ \theta\neq\alpha) \tag{1.18}$$

and draw from (1.17), (1.14), after an elementary computation, that

$$\vartheta_1(\theta, \alpha)=\frac{1-\nu}{\pi\mu}\left[\log\left|\frac{\cos\theta-\cos\alpha}{\cos\theta+\cos\alpha}\right|+2\cos\theta\cos\alpha\right] \qquad (0\leq\theta\leq\frac{\pi}{2}\ ,\ 0\leq\alpha<\frac{\pi}{2}\ ,\ \theta\neq\alpha). \tag{1.19}$$

Further, we set

$$\cos\theta=t\ \ (0\leq\theta\leq\frac{\pi}{2})\quad ,\quad \cos\theta_o=t_o\quad ,\quad \cos\theta_*=t_*\quad ,\quad \cos\alpha=\tau\ \ (0\leq\alpha<\frac{\pi}{2}), \tag{1.20}$$

$$k=\frac{1-\nu}{\pi\mu}\quad ,\quad q(t)=\frac{p(\theta(t))}{\sqrt{1-t^2}} \qquad (t_o<t<1)\ . \tag{1.21}$$

Finally, with reference to (1.4) and (1.20), we write

$$\underset{\sim}{w}(t)=\underset{\sim}{v}(\theta(t)) \qquad (0\leq t\leq 1)\ . \tag{1.22}$$

The first of (1.13), because of (1.18) to (1.22), now yields

$$w_1(t)=v_1(\theta(t))=kr_o\int_{t_o}^{1}\left[\log\left|\frac{t-\tau}{t+\tau}\right|+2t\tau\right]q(\tau)d\tau \qquad (0\leq t\leq 1)\ . \tag{1.23}$$

Moreover, by virtue of (1.23) and (1.20), the displacement boundary condition (1.6) at present furnishes

$$k\int_{t_o}^{1}\left[\log\left|\frac{t-\tau}{t+\tau}\right|+2t\tau\right]q(\tau)d\tau=t_*-t \qquad (t_o\leq t\leq 1)\ , \tag{1.24}$$

while the separation condition (1.8) takes the form

$$\frac{w_1(t)}{r_o}=k\int_{t_o}^{1}\left[\log\left|\frac{t-\tau}{t+\tau}\right|+2t\tau\right]q(\tau)d\tau\leq t_*-t \qquad (0\leq t\leq t_o)\ . \tag{1.25}$$

In addition, conditions (1.9), (1.10), because of (1.12), (1.20), and (1.21), lead to

$$q(t)\geq 0 \quad (t_0<t<1) \quad , \quad 2r_0\int_{t_0}^{1} q(t)dt = P \ . \tag{1.26}$$

Consequently, the original problem is equivalent to that of determining the unknown function q and the parameters t_0, t_* from the singular integral equation (1.24), subject to the side conditions (1.25), (1.26).

We now aim at a further reduction of the problem at hand. For this purpose we introduce an auxiliary constant through

$$a=1+2k\int_{t_0}^{1}\tau q(\tau)d\tau \tag{1.27}$$

and note that (1.24), (1.27) imply the pair of equations

$$k\int_{t_0}^{1}\log\left|\frac{t-\tau}{t+\tau}\right|q(\tau)d\tau = \begin{cases} t_*-at & (t_0<t<1) \\ -t_*-at & (-1<t<-t_0) \ , \end{cases} \tag{1.28}$$

in which the lower equality is an immediate consequence of the upper. Next, we define a function φ by means of

$$\varphi(t)=q(t) \quad (t_0<t<1) \quad , \quad \varphi(t)=-q(-t) \quad (-1<t<-t_0) \tag{1.29}$$

and call L the open point set given by

$$L=(-1,-t_0)\cup(t_0,1) \ . \tag{1.30}$$

Then (1.28) passes over into

$$k\int_{L}\log|t-\tau|\varphi(\tau)d\tau=\psi(t) \text{ for all } t \text{ on } L \ , \tag{1.31}$$

with

$$\psi(t)=t_*-at \quad (t_0<t<1) \quad , \quad \psi(t)=-t_*-at \quad (-1<t<-t_0) \ . \tag{1.32}$$

We suppose henceforth that

$$p(\theta)=O(|\theta-\theta_o|^{-c}) \text{ as } \theta\to\theta_o \quad (c<1) \;, \tag{1.33}$$ [1]

which assures the previously assumed integrability[2] of p on $[0, \theta_o]$. Further, in view of the smoothness of p on $[0, \theta_o)$ and because of (1.20), (1.21), (1.29), (1.30), (1.33), φ is continuously differentiable on L and obeys

$$\varphi(t)=O([1-t^2]^{-\frac{1}{2}}) \text{ as } t\to\pm 1 \quad , \quad \varphi(t)=O([t^2-t_o^2]^{-c}) \text{ as } t\to\pm t_o \;. \tag{1.34}$$

At this stage we introduce an analytic function, the real part of which coincides on L with the left-hand member in (1.31). Thus consider

$$f(\zeta)=k\int_L \log(\zeta-\tau)\varphi(\tau)d\tau \text{ for all } \zeta \text{ on } \bar{S}^+ \;, \tag{1.35}$$

where S^+ is the open upper half of the complex ζ-plane, while $\log(\zeta-\tau)$, for fixed real τ, is to be real for every ζ on (τ, ∞) and analytic on the entire ζ-plane cut along $(-\infty, \tau]$. By virtue of (1.34) and the regularity of φ on L, the function f so defined is analytic on the half-plane S^+ and continuous on its closure $\bar{S}^+$. Also, from (1.35),

$$f'(\zeta)=k\int_L \frac{\varphi(\tau)}{\zeta-\tau}d\tau \text{ for all } \zeta \text{ on } S^+ \;. \tag{1.36}$$

Now let M be the complement of $\bar{L}$ with respect to the real axis, i.e.,

$$M=(-\infty, -1)\cup(-t_o, t_o)\cup(1, \infty) \;, \tag{1.37}$$

and adopt the notation

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[1] Throughout this paper the order-of-magnitude symbols "O" and "o" are used in their standard mathematical connotation.

[2] Recall (1.12) and the regularity assumptions on s_1 introduced prior to (1.10).

$$g=f' \text{ on } S^+ \quad , \quad g^+(t)=\lim_{\zeta\to t} g(\zeta) \text{ for all } t \text{ on } L\cup M \ . \tag{1.38}$$

According to (1.35), (1.31),

$$\mathrm{Re}\{f\}=\psi \text{ on } L \tag{1.39}$$

and thus from (1.32), (1.36), (1.38),

$$\mathrm{Re}\{g^+\}=-a \text{ on } L \quad , \quad \mathrm{Im}\{g^+\}=0 \text{ on } M \ . \tag{1.40}$$[1]

Further, by virtue of (1.36), (1.38), and Plemelj's theorem (see Muskhelishvili [11], p. 42),

$$g^+(t)=-i\pi k\varphi(t)+k\int_L \frac{\varphi(\tau)}{t-\tau}\,d\tau \text{ for all } t \text{ on } L \ , \tag{1.41}$$

provided the integral in (1.41) is interpreted in the sense of its Cauchy principal value. Consequently,

$$\varphi=-\frac{1}{\pi k}\mathrm{Im}\{g^+\} \text{ on } L \ . \tag{1.42}$$

According to its definition (1.29), φ is an odd function. Hence (1.36) and the first of (1.38) give

$$g(\zeta)=k\int_L \varphi(\tau)\left[\frac{1}{\zeta-\tau}-\frac{1}{\zeta}\right]d\tau=k\int_L \frac{\tau\varphi(\tau)}{\zeta(\zeta-\tau)}\,d\tau \text{ for all } \zeta \text{ on } S^+ \tag{1.43}$$

so that

$$g(\zeta)=O(\zeta^{-2}) \text{ as } \zeta\to\infty \ . \tag{1.44}$$

Finally, in view of (1.42), the order conditions (1.34) are guaranteed to be met by φ if g satisfies

$$g(\zeta)=O\left(|1-\zeta^2|^{-\frac{1}{2}}\right) \text{ as } \zeta\to\pm 1 \quad , \quad g(\zeta)=O\left(|\zeta^2-t_o^2|^{-c}\right) \text{ as } \zeta\to\pm t_o \ , \tag{1.45}$$

where once again $c<1$.

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[1] Recall that k and φ are real-valued.

It is clear from what preceded that the solution of the singular integral equation (1.31), (1.32) for the unknown function φ, subject to (1.34), has now been reduced to the task of finding a function g that is analytic on S^+, satisfies the boundary conditions (1.40), and meets the regularity requirements (1.44), (1.45). The complete solution of this mixed Hilbert problem for the half-plane is known[1] and given by the restriction to S^+ of the function g defined through

$$g(\zeta) = \frac{a\zeta^2+b}{\sqrt{Q(\zeta)}} - a \text{ for all } \zeta \text{ not on } \overline{L}, \tag{1.46}$$

where

$$\sqrt{Q(\zeta)} = \sqrt{\left(\zeta^2-1\right)\left(\zeta^2-t_o^2\right)} \tag{1.47}$$

is analytic on the entire ζ-plane cut along $\overline{L}$ and is real and positive on $(1, \infty)$. The constant a appearing in (1.46) was initially introduced in (1.27) and is as yet indeterminate; b is another still arbitrary real constant.

From (1.46), (1.47) and the second of (1.38) follows

$$g^+(t) = -i\frac{at^2+b}{\sqrt{\left(1-t^2\right)\left(t^2-t_o^2\right)}} - a \quad (t_o<t<1) . \tag{1.48}$$

Therefore, bearing in mind (1.42), (1.29), one has

$$q(t) = \varpi(t) = -\frac{1}{\pi k}\operatorname{Im}\{g^+(t)\} = \frac{1}{\pi k}\,\frac{at^2+b}{\sqrt{\left(1-t^2\right)\left(t^2-t_o^2\right)}} \quad (t_o<t<1) , \tag{1.49}$$

which, together with the second of (1.21), yields

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[1] See Muskhelishvili [11], art. 95. The complete solution of the general mixed Hilbert problem for the half-plane is apparently originally due to Signorini [12] (1916). Cf. also Keldysh and Sedov [13].

$$p(\theta(t)) = \frac{1}{\pi k} \frac{at^2+b}{\sqrt{t^2-t_o^2}} \qquad (t_o < t \leq 1) \; . \tag{1.50}$$

It remains to determine the constants a,b and the parameters t_o, t_* in such a way that the function q given by (1.49) satisfies the integral equation (1.24) together with the three side conditions contained in (1.25), (1.26). This is accomplished in the next section, which — as a by-product — furnishes a verification of the solution (1.13), (1.50) to the original problem.

2. Adaptation of solution to the auxiliary Hilbert problem. Verification of solution to the original problem.

Our initial objective here is to derive conditions on the still arbitrary constants a,b, sufficient to insure that the function q given by (1.49) satisfies the integral equation (1.24) for every choice of t_o and t_* $(0 < t_o, t_* < 1)$. To this end it evidently suffices to determine a,b such that

$$w_1(t) = r_o(t_* - t) \qquad (t_o \leq t \leq 1) \; , \tag{2.1}$$

with w_1 defined by (1.23) and (1.49). We now deduce from (1.23), (1.49) a representation of w_1 that is suitable for the present purpose and, for future convenience, at once consider the entire range [0, 1].

For each fixed t in [0, 1], let $\log[(\tau-t)/(\tau+t)]$ be taken as analytic in the complex τ-plane cut along $[-t, t]$ and real-valued on (t, ∞). Next, for all t in [0, 1] and every τ in the cut complex plane just described, let

$$\Lambda(\tau, t) = \left[\log \frac{\tau-t}{\tau+t} + 2t\tau\right] g(\tau) \; , \tag{2.2}$$

where g is the function defined in (1.46), (1.47). Further, let $\Lambda^+(\tau, t)$ and $\Lambda^-(\tau, t)$ stand for the respective limits of $\Lambda(\tau, t)$ as τ approaches the cut from above or below. It is easily confirmed with the aid of (2.2), (1.23), (1.48), (1.49) that

$$w_1(t)=-\frac{r_o}{\pi}\mathrm{Im}\left\{\int_{t_o}^{1}\Lambda^+(\tau,t)d\tau\right\}-r_o a(t-t_o)H(t-t_o)\qquad(0\le t\le 1)\ , \tag{2.3}$$

where H is the Heaviside unit step-function given by

$$H(t)=1\quad(t\ge 0)\quad,\qquad H(t)=0\quad(t<0)\ . \tag{2.4}$$

The required imaginary part of the integral in (2.3) may be evaluated by means of a contour integration. Thus, let $\Gamma(\eta,\rho)$ be the closed contour in the complex τ-plane depicted in Figure 3. Then, evidently,

$$\oint_{\Gamma(\eta,\rho)}\Lambda(\tau,t)d\tau=0\qquad(0\le t\le 1)\ . \tag{2.5}$$

Moreover, from the definition of Λ through (2.2), (1.46), (1.47) one finds, for every t in [0, 1],

$$\Lambda(\tau,t)=\left[\left(1+t_o^2\right)a+2b\right]\frac{t}{\tau}+o(\tau^{-1})\quad\text{as}\quad\tau\to\infty\ , \tag{2.6}$$

$$\left.\begin{aligned}&\mathrm{Im}\{\Lambda^-(\tau,t)\}=-\mathrm{Im}\{\Lambda^+(\tau,t)\}\qquad(0\le\tau<1\ ,\ \tau\ne t\ ,\ \tau\ne t_o)\ ,\\&\mathrm{Im}\{\Lambda^+(\tau,t)\}=\pi H(t-\tau)g(\tau)\qquad(0\le\tau<t_o)\ ,\end{aligned}\right\} \tag{2.7}$$

where

$$g(\tau)=-\frac{a\tau^2+b}{\sqrt{\left(1-\tau^2\right)\left(t_o^2-\tau^2\right)}}-a\qquad(0\le\tau<t_o)\ ; \tag{2.8}$$

in addition,

$$\mathrm{Im}\left\{\int_{-i\infty}^{i\infty}\Lambda(\tau,t)d\tau\right\}=0 \tag{2.9}$$

since $\Lambda(\tau,t)$ $(0\le t\le 1)$ is purely imaginary for imaginary values of τ. Bearing in mind (2.6), (2.7), (2.9), and passing to the limit in (2.5) as $\eta\to 0$ and $\rho\to\infty$, one infers that

$$\mathrm{Im}\left\{\int_{t_o}^{1} \Lambda^+(\tau, t)d\tau\right\} = -\left[\left(1+t_o^2\right)a+2b\right]\frac{\pi t}{2} - \pi\int_0^{t_o} H(t-\tau)g(\tau)d\tau \qquad (0\le t\le 1) , \qquad (2.10)$$

with $g(\tau)$ given by (2.8). Finally, combining (2.10), (2.8), (2.3), one arrives at

$$w_1(t) = -r_o\int_0^{t_o} H(t-\tau)\left[\frac{a\tau^2+b}{\sqrt{(1-\tau^2)\left(t_o^2-\tau^2\right)}}+a\right]d\tau$$

$$+\left[\left(1+t_o^2\right)a+2b\right]\frac{r_o t}{2} - r_o a(t-t_o)H(t-t_o) \qquad (0\le t\le 1) , \qquad (2.11)$$

where H is the unit step-function defined in (2.4).

The integral appearing in (2.11) is expressible in terms of elliptic integrals. For this purpose we recall Legendre's standard form of the incomplete elliptic integrals of the first and second kind

$$\left.\begin{aligned} F(\varkappa, \beta) &= \int_0^{\sin\beta} \frac{d\tau}{\sqrt{(1-\tau^2)(1-\varkappa^2\tau^2)}} \qquad \left(0\le\beta\le\frac{\pi}{2} , \quad 0\le\varkappa<1\right) , \\ E(\varkappa, \beta) &= \int_0^{\sin\beta} \sqrt{\frac{1-\varkappa^2\tau^2}{1-\tau^2}}\, d\tau \qquad \left(0\le\beta\le\frac{\pi}{2} , \quad 0\le\varkappa<1\right) , \end{aligned}\right\} \qquad (2.12)$$

in which $\varkappa$ is the modulus, and set

$$t = t_o\sin\beta \quad , \quad 0\le\beta\le\frac{\pi}{2} \quad (0\le t\le t_o) . \qquad (2.13)$$

From (2.11), (2.4), (2.12), (2.13) one infers, after elementary manipulations, that

$$\left.\begin{aligned} w_1(t) &= -r_o\left\{(a+b)F(t_o, \beta) - aE(t_o, \beta) + \left[\left(1-t_o^2\right)a-2b\right]\frac{t}{2}\right\} \qquad (0\le t\le t_o) , \\ w_1(t) &= -r_o\left\{(a+b)K - aE + \left[\left(1-t_o^2\right)a-2b\right]\frac{t}{2}\right\} \qquad (t_o<t\le 1) , \end{aligned}\right\} \qquad (2.14)$$

provided K and E designate the complete elliptic integrals

$$K \equiv K(t_o) = F\left(t_o, \frac{\pi}{2}\right) \quad , \quad E \equiv E(t_o) = E\left(t_o, \frac{\pi}{2}\right) . \tag{2.15}$$

It is clear from the second of (2.14) that (2.1) holds if and only if the two constants a and b satisfy the pair of linear algebraic equations

$$\left(1-t_o^2\right)a - 2b = 2 \quad , \quad a(E-K) - bK = t_* . \tag{2.16}$$

Accordingly, the function q represented by (1.49) satisfies the integral equation (1.24) if and only if a and b conform to (2.16).

We turn now to necessary consequences of the two inequalities supplied by (1.25) and the first of (1.26). On rewriting (1.49) in the form

$$q(t) = \frac{1}{\pi k}\sqrt{\frac{t^2-t_o^2}{1-t^2}}\left[\frac{at_o^2+b}{t^2-t_o^2} + a\right] \quad (t_o < t < 1) \tag{2.17}$$

it is at once apparent that the first of (1.26) requires

$$at_o^2 + b \geq 0 . \tag{2.18}$$

On the other hand, the inequality (1.25), because of (2.11), (2.12), (2.15), (2.16), is found to be equivalent to

$$\int_t^{t_o} \frac{a\tau^2+b}{\sqrt{(1-\tau^2)(\tau^2-t_o^2)}}\, d\tau \leq 0 \quad (0 \leq t \leq t_o) . \tag{2.19}$$

We show next that (2.19), in turn, implies

$$at_o^2 + b \leq 0 . \tag{2.20}$$

Indeed, suppose contrariwise that $at_o^2+b > 0$. Then there is a value of t in $[0, t_o)$ such that $a\tau^2+b > 0$ for all τ in $[t, t_o]$, which contradicts (2.19). Hence (2.20) holds. But (2.18), (2.20) give

$$at_o^2 + b = 0 . \tag{2.21}$$

Finally, from (2.21), (2.16) follows

$$a=\frac{2}{1+t_o^2} \quad , \quad b=-\frac{2t_o^2}{1+t_o^2} \quad , \quad t_*=\frac{2}{1+t_o^2}\left[E-\left(1-t_o^2\right)K\right] . \tag{2.22}$$

Conversely, (2.22) are also <u>sufficient</u> to ensure that q satisfy the inequalities in (1.25), (1.26). To see this, observe that (2.22), (2.17) imply

$$q(t)=\frac{2}{\pi k\left(1+t_o^2\right)}\sqrt{\frac{t^2-t_o^2}{1-t^2}} \quad (t_o \leq t<1) , \tag{2.23}$$

so that $q \geq 0$ on $[t_o, 1)$; further, (2.22) evidently guarantee (2.19), which is equivalent to (1.25).

In view of (1.50), the actual pressure p may be written as

$$p(\theta(t))=\frac{1}{\pi k}\sqrt{t^2-t_o^2}\left[\frac{at_o^2+b}{t^2-t_o^2}+a\right] \quad (t_o<t\leq 1) . \tag{2.24}$$

Therefore the relation (2.21) is at the same time necessary and sufficient for p to be bounded. As is now apparent, this regularity property of the contact pressure is a consequence of the separation condition (1.8) and of the condition of unilateral constraint (1.9): an a priori requirement to the effect that p be bounded — apart from its lack of motivation — would have led to a redundancy in the formulation of the problem. In this connection it is also worth mentioning that (2.21) may alternatively be shown to constitute a necessary condition that the deformed boundary of the roller cross-section possess a continuously turning tangent at the endpoints of its contact segments, as is readily confirmed with the aid of (2.11).

Equations (2.22), (2.23) involve the single parameter t_o, which needs to be determined from the load condition given by the second of (1.26). The latter, in conjunction with (2.23) and the first of (1.21), yields

$$P = \frac{4r_o\mu}{(1-\nu)\left(1+t_o^2\right)} \int_{t_o}^{1} \sqrt{\frac{\tau^2 - t_o^2}{1-\tau^2}}\, d\tau \ , \tag{2.25}$$

whence, on account of (2.12), (2.15),

$$P = \frac{4r_o\mu}{(1-\nu)\left(1+t_o^2\right)} \left[E' - t_o^2 K'\right] . \tag{2.26}$$

Here K' and E' are the complete elliptic integrals of the first and second kind for the modulus complementary with respect to t_o, i.e.,

$$K' = K(t_o') \quad , \quad E' = E(t_o') \quad , \quad t_o' = \sqrt{1-t_o^2} \ . \tag{2.27}$$

The numerical computation of t_o from a given P on the basis of (2.26) presents no difficulties. An asymptotic inversion of (2.26) in elementary form, adequate for all practical purposes, will be carried out in the succeeding section.

3. Results for the contact pressure and the boundary deformations of the roller. Asymptotic estimates.

The final result for the distribution of the contact pressure is immediate from (2.24), (2.21), the first of (2.22), (1.20), and the first of (1.21):

$$p(\theta) = \frac{2\mu}{(1-\nu)\left(1+t_o^2\right)} \sqrt{t^2 - t_o^2} \quad , \quad t = \cos\theta \quad , \quad t_o = \cos\theta_o \qquad (0 \le \theta \le \theta_o) \ . \tag{3.1}$$

This formula is elementary but for the transcendental dependence of t_o upon the total load P, implicit in (2.26). In particular, (3.1) yields for the maximum pressure

$$p_o \equiv p(0) = \frac{2\mu\sqrt{1-t_o^2}}{(1-\nu)\left(1+t_o^2\right)} \quad , \quad t_o = \cos\theta_o \ . \tag{3.2}$$

Further, the diametral compression Δ of the roller (see Figure 1), because of (1.7), (1.20), and (2.22), at once becomes

$$\frac{\Delta}{r_o}=2\left\{1-\frac{2}{1+t_o^2}\left[E-\left(1-t_o^2\right)K\right]\right\} \quad , \quad t_o=\cos\theta_o \ , \tag{3.3}$$

in which K and E, as before, are the first and second-kind complete elliptic integrals for the modulus t_o.

Another quantity of direct physical interest is the width ℓ of the contact zone (see Figure 1). By virtue of (1.4), (1.20), (1.22), one evidently has

$$\ell=2\left[r_o\sin\theta_o+v_2(\theta_o)\right]=2\left[r_o\sqrt{1-t_o^2}+w_2(t_o)\right] \ , \tag{3.4}$$

so that the determination of ℓ hinges on a knowledge of $w_2(t_o)$. We therefore postpone the calculation of the contact width and turn first to the determination of the complete boundary displacements $\underset{\sim}{v}(\theta)$ $(0\le\theta\le\pi/2)$, which are of interest in themselves.

The surface displacement v_1 was in effect established in the course of the analysis contained in Section 2. In fact, substitution for a and b from (2.22) into (2.14) leads to the closed elliptic-integral representation

$$\left.\begin{aligned} v_1(\theta(t))=w_1(t)&=-\frac{2r_o}{1+t_o^2}\left[\left(1-t_o^2\right)F(t_o,\beta)-E(t_o,\beta)\right]-t \qquad (0\le t\le t_o) \ , \\ v_1(\theta(t))=w_1(t)&=-\frac{2r_o}{1+t_o^2}\left[\left(1-t_o^2\right)K-E\right]-t \qquad (t_o\le t\le 1) \ , \end{aligned}\right\} \tag{3.5}$$

with $t_o=\cos\theta_o$ and $F(t_o,\beta)$, $E(t_o,\beta)$, K, E accounted for in (2.12), (2.13), (2.15).

Aiming at the analogous result for $v_2(\theta)$ $(0\le\theta\le\pi/2)$, we note that (1.17), (1.14), (1.18), after a routine computation, furnish

$$\overset{\circ}{v}_2(\theta,\alpha) = -\frac{1}{2\pi\mu}\left[(1-2\nu)\pi H(\theta-\alpha) - 4(1-\nu)\sin\theta\cos\alpha\right] \quad \left(0 \le \theta \le \frac{\pi}{2}\ ,\ 0 \le \alpha < \frac{\pi}{2}\ ,\ \theta \ne \alpha\right)\ , \tag{3.6}$$

where H is the step function introduced in (2.4). From (3.6), (1.13), (1.20), (1.21), (1.22) now follows

$$v_2(\theta(t)) = w_2(t) = 2kr_o\sqrt{1-t^2}\int_{t_o}^{1}\tau q(\tau)d\tau - \frac{(1-2\nu)r_o}{2\mu}\int_{t_o}^{1}H(\tau-t)q(\tau)d\tau \qquad (0 \le t \le 1)\ , \tag{3.7}$$

q being given by (2.23). Hence the first integral in (3.7) is elementary, whereas the second is expressible in terms of elliptic integrals. Thus, recalling (2.27) and putting

$$\sqrt{1-t^2} = t_o'\sin\beta'\quad ,\quad 0 \le \beta' \le \frac{\pi}{2}\quad ,\quad t_o' = \sqrt{1-t_o^2}\qquad (t_o \le t \le 1)\ , \tag{3.8}$$

one finds after suitable changes of the variable of integration in (3.7):

$$\left.\begin{aligned} & v_2(\theta(t)) = w_2(t) = \\ & \frac{r_o\left(1-t_o^2\right)}{1+t_o^2}\sqrt{1-t^2} - \frac{r_o(1-2\nu)}{(1-\nu)\left(1+t_o^2\right)}\left[E(t_o',\beta') - t_o^2F(t_o',\beta')\right] \quad (t_o \le t \le 1)\ , \\ & v_2(\theta(t)) = w_2(t) = \frac{r_o\left(1-t_o^2\right)}{1+t_o^2}\sqrt{1-t^2} - \frac{r_o(1-2\nu)}{(1-\nu)\left(1+t_o^2\right)}\left[E' - t_o^2K'\right] \quad (0 \le t \le t_o)\ , \end{aligned}\right\} \tag{3.9}$$

if F and E are the incomplete elliptic integrals cited in (2.12), while K' and E' stand for complete integrals defined by (2.27). From (3.9) and (3.4) one obtains for the contact width

$$\frac{\ell}{r_o} = \frac{4\sqrt{1-t_o^2}}{1+t_o^2} - \frac{2(1-2\nu)}{(1-\nu)\left(1+t_o^2\right)}\left[E' - t_o^2K'\right]\quad ,\quad t_o = \cos\theta_o\ . \tag{3.10}$$

It is possible to deduce from (1.13), (1.17), and (3.1) an elliptic integral representation for the entire displacement field of the roller. In the interest of brevity we shall not pursue this cumbersome task and shall proceed instead to the derivation of convenient asymptotic estimates for the physically most significant elements of the solution to the contact problem under consideration.

In preparation for the present objective we set

$$\epsilon = t_o' = \sqrt{1-t_o^2} = \sin\theta_o \tag{3.11}$$

and note from familiar asymptotic properties of complete elliptic integrals[1] that

$$\left.\begin{aligned} K' = K(\epsilon) &= \frac{\pi}{2}\left[1+\frac{\epsilon^2}{4}+\frac{9\epsilon^4}{64}\right]+O(\epsilon^6) \quad \text{as} \quad \epsilon\to 0 \ , \\ E' = E(\epsilon) &= \frac{\pi}{2}\left[1-\frac{\epsilon^2}{4}-\frac{3\epsilon^4}{64}\right]+O(\epsilon^6) \quad \text{as} \quad \epsilon\to 0 \ ; \end{aligned}\right\} \tag{3.12}$$

$$\left.\begin{aligned} K = K(t_o) &= \lambda + \frac{\lambda-1}{4}\epsilon^2 + O(\lambda\epsilon^4) \quad \text{as} \quad \epsilon\to 0 \ , \\ E = E(t_o) &= 1 + \frac{2\lambda-1}{4}\epsilon^2 + \frac{12\lambda-13}{64}\epsilon^4 + O(\lambda\epsilon^6) \quad \text{as} \quad \epsilon\to 0 \ , \end{aligned}\right\} \tag{3.13}$$

where

$$\lambda = \log\frac{4}{\epsilon} = \log\frac{4}{\sqrt{1-t_o^2}} \ . \tag{3.14}$$

From (2.26), (3.11), (3.12) easily follows

$$P = \frac{\pi r_o \mu}{2(1-\nu)}\left[\epsilon^2+\frac{5}{8}\epsilon^4+O(\epsilon^6)\right] \quad \text{as} \quad \epsilon\to 0 \ , \tag{3.15}$$

while (3.2), (3.11) furnish

$$\frac{p_o}{\mu} = \frac{1}{1-\nu}\left[\epsilon+\frac{1}{2}\epsilon^3+O(\epsilon^5)\right] \quad \text{as} \quad \epsilon\to 0 \ . \tag{3.16}$$

Further, (3.3), (3.11), (3.13) lead to

[1] See, for example, Jahnke and Emde [14], p. 73.

$$\frac{\Delta}{r_o} = \frac{2\lambda-1}{2}\epsilon^2 + \frac{20\lambda-11}{32}\epsilon^4 + O(\lambda\epsilon^6) \text{ as } \epsilon\to 0 \ . \tag{3.17}$$

The corresponding estimate for ℓ/r_o is immediate from (3.15), (3.16) since

$$\frac{\ell}{r_o} = \frac{2(1-\nu)}{\mu}p_o - \frac{1-2\nu}{2r_o\mu}P \tag{3.18}$$

as a consequence of (3.10), (2.26), and (3.2).

It is clear from (3.11) and the meaning of θ_o that ϵr_o is the half-length of the chord subtended by each of the two circular arcs of ∂R that are brought into contact with the rigid plates upon deformation of the roller (see Figure 1). Hence ϵ is not known directly from the data of the problem. In contrast, the dimensionless parameter

$$\delta = \sqrt{\frac{2(1-\nu)P}{\pi r_o\mu}} \tag{3.19}$$

is free from this objection. For this reason we now use the above results to obtain estimates of p_o/μ, Δ/r_o, and ℓ/r_o in terms of δ.

To this end observe from (3.15), (3.19) that

$$\delta^2 = \epsilon^2 + \frac{5}{8}\epsilon^4 + O(\epsilon^6) \text{ as } \epsilon\to 0 \ , \tag{3.20}$$

whence

$$\epsilon^2 = \delta^2 - \frac{5}{8}\delta^4 + O(\delta^6) \text{ as } \delta\to 0 \ . \tag{3.21}$$

Equation (3.21) in conjunction with (3.14), (3.16), (3.17), (3.18), (3.19), after trivial computations, yield the practically useful asymptotic results

$$\frac{(1-\nu)p_o}{\mu} = \left[\frac{2(1-\nu)P}{\pi r_o\mu}\right]^{1/2} + \frac{3}{16}\left[\frac{2(1-\nu)P}{\pi r_o\mu}\right]^{3/2} + O(\delta^5) \ , \tag{3.22}$$

$$\frac{\Delta}{r_o} = \frac{(1-\nu)P}{\pi r_o\mu}\left[\log\frac{8\pi r_o\mu}{(1-\nu)P} - 1\right] + \frac{9(1-\nu)^2P^2}{8\pi^2 r_o^2\mu^2} + O(\delta^6\log\delta) \ , \tag{3.23}$$

$$\frac{\ell}{r_o}=\left[\frac{8(1-\nu)P}{\pi r_o \mu}\right]^{1/2}-\frac{(1-2\nu)P}{2r_o\mu}+O(\delta^3) \ , \tag{3.24}$$

all of which hold rigorously in the limit as δ, given by (3.19), tends to zero.

The preceding estimates for the maximum contact pressure, the diametral compression, and the contact width may evidently be refined to any desired degree of accuracy. However, since δ is bound to be quite small compared to unity in any actual physical example[1], the two-term asymptotic expansions (3.22), (3.23), (3.24) are apt to be more than adequate. The leading term in (3.22) and its counterpart in (3.24) coincide precisely with the predictions of the Hertz theory[2] for the maximum contact pressure and the contact width, respectively. The corresponding second terms represent corrections to the Hertz theory, which are seen to be rather insignificant. The dominating term in the asymptotic expansion (3.23) of the exact diametral compression, in turn, is identical with the analogous formula obtained through the appropriate specialization of the approximate results due to Loo [8] and Johnson [7].

4. Closed elementary representation of the stress field.

The computation of the stress distribution in the roller is most expeditiously accomplished by recourse to (1.14), (1.15), (1.16), and the second of (1.13), in which p is now given by (3.1). Thus,

$$\left.\begin{aligned} &\sigma_{11}(\underset{\sim}{x})+\sigma_{22}(\underset{\sim}{x})=4\mathrm{Re}\{\Phi(z)\} \ , \\ &\sigma_{22}(\underset{\sim}{x})-\sigma_{11}(\underset{\sim}{x})+2i\sigma_{12}(\underset{\sim}{x})=2[\bar{z}\Phi'(z)+\Psi(z)] \ , \end{aligned}\right\} \tag{4.1}$$

[1] Thus for a steel roller $\delta \doteq 2\times10^{-3}$ if the maximum contact pressure is 3×10^4 psi.

[2] See, for example, [3], p. 382.

for all $\underset{\sim}{x}$ in $\overline{R}$, where

$$\left.\begin{aligned} \Phi(z)&=r_o\int_0^{\theta_o}\dot{\Phi}(z,\mathfrak{z}(\alpha))p(\alpha)d\alpha\ ,\\ \Psi(z)&=r_o\int_0^{\theta_o}\dot{\Psi}(z,\mathfrak{z}(\alpha))p(\alpha)d\alpha\ ,\end{aligned}\right\}\tag{4.2}$$

while Φ' denotes the derivative of Φ. Moreover, because of (1.14), (1.16), the last of (1.20) and the second of (1.21), one readily confirms that the generating complex potentials (4.2) may be written as

$$\Phi(z)=-\frac{1}{\pi}[\Theta(z)-A]\quad,\quad \Psi(z)=\frac{1}{\pi}[\Theta(z)+\Omega'(z)]\ ,\tag{4.3}$$

provided

$$\left.\begin{aligned}\Theta(z)&=r_o\int_{t_o}^{1}\frac{r_o\tau-z}{r_o^2-2r_oz\tau+z^2}q(\tau)d\tau+r_o\int_{t_o}^{1}\frac{r_o\tau+z}{r_o^2+2r_oz\tau+z^2}q(\tau)d\tau\ ,\\ \Omega(z)&=r_o^2\int_{t_o}^{1}\frac{2r_o\tau^2-z\tau-r_o}{r_o^2-2r_oz\tau+z^2}q(\tau)d\tau-r_o^2\int_{t_o}^{1}\frac{2r_o\tau^2+z\tau-r_o}{r_o^2+2r_oz\tau+z^2}q(\tau)d\tau\ ,\end{aligned}\right\}\tag{4.4}$$ [1]

$$A=\int_{t_o}^{1}\tau q(\tau)d\tau\ ,\tag{4.5}$$

and q is as in (2.23). The integral defining A is elementary. Indeed, (2.23), (4.5), and the first of (1.21) give

$$A=\frac{\pi\mu\left(1-t_o^2\right)}{2(1-\nu)\left(1+t_o^2\right)}\ .\tag{4.6}$$

On the other hand, (4.4) may be put into a more convenient form by use of

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[1] Ω' in (4.3) is the derivative of Ω.

(1.29), (1.30). In this manner, and with the aid of (2.23), (1.21), one arrives at

$$\left.\begin{aligned} \Theta(z) &= r_o \int_L \frac{r_o\tau - z}{r_o^2 - 2r_o z\tau + z^2}\varphi(\tau)d\tau \ , \\ \Omega(z) &= r_o^2 \int_L \frac{2r_o\tau^2 - z\tau - r_o}{r_o^2 - 2r_o z\tau + z^2}\varphi(\tau)d\tau \ , \end{aligned}\right\} \tag{4.7}$$

where

$$\varphi(\tau) = c\sqrt{\frac{\tau^2 - t_o^2}{1-\tau^2}} \quad (t_o \le \tau < 1) \quad , \quad \varphi(\tau) = -c\sqrt{\frac{\tau^2 - t_o^2}{1-\tau^2}} \quad (-1 < \tau \le -t_o) \ , \tag{4.8}$$

and

$$c = \frac{2\mu}{(1-\nu)\left(1+t_o^2\right)} \ . \tag{4.9}$$

In preparation for a contour-integral evaluation of the auxiliary complex potentials Θ, Ω we now introduce a pair of functions χ_1, χ_2 defined, for all z in $\overline{R}$ and all τ in the complex τ-plane cut along the closure of L, by means of

$$\left.\begin{aligned} \chi_1(\tau, z) &= \frac{r_o\tau - z}{r_o^2 - 2r_o z\tau + z^2}\sqrt{\frac{\tau^2 - t_o^2}{\tau^2 - 1}} \ , \\ \chi_2(\tau, z) &= \frac{2r_o\tau^2 - z\tau - r_o}{r_o^2 - 2r_o z\tau + z^2}\sqrt{\frac{\tau^2 - t_o^2}{\tau^2 - 1}} \ . \end{aligned}\right\} \tag{4.10}$$

The square-root in (4.10) is understood to be analytic in the cut τ-plane, as well as real and positive on $(1, \infty)$. Let $\chi_j^+(\tau, z)$ and $\chi_j^-(\tau, z)$ $(j=1, 2)$ refer to the respective limits of $\chi_j(\tau, z)$ as τ approaches the cut L from above or below. Then (4.10), (4.7), (4.9) give, for all z in $\overline{R}$,

$$\Theta(z)=ir_o c\int_L \chi_1^+(\tau,z)d\tau \quad , \quad \Omega(z)=ir_o^2 c\int_L \chi_2^+(\tau,z)d\tau \ . \tag{4.11}$$

Assuming temporarily[1] $z\neq 0$, we note that $\chi_j(\tau,z)$ $(j=1,2)$ have simple poles at $\tau=\hat{\tau}$, provided

$$\hat{\tau}=\frac{r_o^2+z^2}{2r_o z} \ . \tag{4.12}$$

Consequently, if $\Gamma(\eta,\rho)$ is the closed contour in the complex τ-plane shown in Figure 4, equations (4.10) and the residue theorem permit the conclusion

$$\oint_{\Gamma(\eta,\rho)} \chi_1(\tau,z)d\tau=-\frac{\pi i}{2r_o z^2}W(z) \quad , \quad \oint_{\Gamma(\eta,\rho)} \chi_2(\tau,z)d\tau=-\frac{\pi i}{2z^3}W(z) \ , \tag{4.13}$$

with

$$W(z)=\sqrt{\left(r_o^2-z^2\right)^2+4\epsilon^2 r_o^2 z^2}=\sqrt{\left(z^2-z_o^2\right)\left(z^2-\bar{z}_o^2\right)} \ . \tag{4.14}$$

Here ϵ has the meaning attached to it in (3.11) and

$$z_o=r_o\exp(i\theta_o) \ . \tag{4.15}$$

Further, W is required to be analytic on R and to have real, positive values on $(-r_o, r_o)$. From their definition (4.10) the functions χ_j are readily found to possess the properties

$$\chi_j^+(\tau,z)=-\chi_j^-(\tau,z) \text{ for all } \tau \text{ on } L \quad (j=1,2) \ , \tag{4.16}$$

$$\left.\begin{aligned}\chi_1(\tau,z)&=-\frac{1}{2z}-\frac{r_o^2-z^2}{4r_o z^2}\,\frac{1}{\tau}+O(\tau^{-2}) \text{ as } \tau\to\infty \ ,\\ \chi_2(\tau,z)&=-\frac{\tau}{z}-\frac{r_o}{2z^2}-\frac{1}{4z^3}\left(r_o^2-z^2+2\epsilon^2 z^2\right)\frac{1}{\tau}+O(\tau^{-2}) \text{ as } \tau\to\infty \ .\end{aligned}\right\} \tag{4.17}$$

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[1] This restriction will eventually be removed. See the remark following (4.20).

Proceeding to the limit in (4.13) as $\eta \to 0$ and $\rho \to \infty$, one finds upon invoking (4.16), (4.17),

$$\left.\begin{aligned} 2\int_L \chi_1^+(\tau, z)d\tau &= -\frac{\pi i}{2r_o z^2}\left[W(z)+z^2-r_o^2\right] , \\ 2\int_L \chi_2^+(\tau, z)d\tau &= -\frac{\pi i}{2z^3}\left[W(z)+z^2-r_o^2\right]+\frac{\pi i\epsilon^2}{z} , \end{aligned}\right\} \quad (4.18)$$

Finally, combining (4.18) with (4.11), one is led to the representations

$$\Theta(z)=\frac{\pi c}{2}\left[z^2 U(z)+\epsilon^2\right] \quad , \quad \Omega(z)=\frac{\pi c}{2} r_o^2 z U(z) , \quad (4.19)$$

in which

$$U(z)=\frac{1}{2z^4}\left[W(z)+(1-2\epsilon^2)z^2-r_o^2\right]=\frac{2\epsilon^2(1-\epsilon^2)}{W(z)-(1-2\epsilon^2)z^2+r_o^2} . \quad (4.20)$$

Note that the first representation of U in (4.20) has a removable singularity at $z=0$, whereas the second is valid for all z in $\overline{R}$.

From (4.20), (4.14) follows, after some manipulation,

$$zU'(z)=2U(z)\left[\frac{r_o^2}{W(z)}-1\right] , \quad (4.21)$$

while (4.3), (4.6), (4.9), (3.11), (4.19), and (4.21) yield the desired complex stress potentials on $\overline{R}$ in the form

$$\left.\begin{aligned} \Phi(z) &= -\frac{c}{2}\left[z^2 U(z)+\frac{\epsilon^2}{2}\right] , \\ \Psi(z) &= \frac{c}{2}\left\{\left[\frac{2r_o^4}{W(z)}+z^2-r_o^2\right]U(z)+\epsilon^2\right\} . \end{aligned}\right\} \quad (4.22)$$

Further, (4.1), (4.22), (4.21), furnish the subsequent results for the cartesian components of stress, which hold for all z in $\overline{R}$:

$$\sigma_{11}(\underset{\sim}{x})=-\frac{c}{2}\mathrm{Re}\left\{\left[\frac{2r_o^2}{W(z)}\left(r_o^2-z\bar{z}\right)+3z^2-r_o^2\right]U(z)\right\}-c\epsilon^2 , \quad (4.23)$$

$$\left.\begin{aligned}\sigma_{22}(\underset{\sim}{x}) &= \frac{c}{2}\,\mathrm{Re}\left\{\left[\frac{2r_o^2}{W(z)}\left(r_o^2 - z\bar{z}\right) - z^2 - r_o^2\right]U(z)\right\}, \\ \sigma_{12}(\underset{\sim}{x}) &= \frac{c}{2}\,\mathrm{Im}\left\{\left[\frac{2r_o^2}{W(z)}\left(r_o^2 - z\bar{z}\right) + z^2 - r_o^2\right]U(z)\right\}.\end{aligned}\right\} \quad \begin{array}{c}(4.23)\\ \text{cont.}\end{array}$$

Here W and U are the elementary functions given by (4.14) and the second equality in (4.20), while the parameters c and ϵ are supplied by (4.9) and (3.11) in terms of $t_o \equiv \cos\theta_o$. Thus the closed representation (4.23) displays an elementary position-dependence of the stress field $\underset{\sim}{\sigma}$. In contrast, the dependence of $\underset{\sim}{\sigma}$ upon the lineal load-density P is transcendental since t_o is related to P in the manner of (2.26). Note that (4.23) assure the continuity of the stress field up to the entire boundary[1].

A brief routine computation now leads from (4.23) to fully explicit formulas for σ_{11}, σ_{22}, σ_{12}, which involve only real-valued position variables. As these formulas are naturally quite lengthy, they will be omitted. We cite instead merely the simple final results for the normal stresses along the diameter of the roller cross-section at right angles to the plate faces. These principal stresses, which are of particular physical interest, follow immediately from (4.23), (4.14), (4.20) because W(z) — and hence also U(z) — is real for real values of z. On setting

$$\xi = \frac{x_1}{r_o}, \quad \omega(\xi) = \sqrt{(1-\xi^2)^2 + 4\epsilon^2\xi^2} \quad (-1 \le \xi \le 1), \tag{4.24}$$

one finds, for $-r_o \le x_1 \le r_o$ and $x_2 = 0$,

[1]Recall that the original formulation of the problem in Section 1 admitted the possibility that $\underset{\sim}{\sigma}$ becomes unbounded at the endpoints of the two contact-arcs.

$$\left.\begin{aligned}\sigma_{11}(\underset{\sim}{x}) &= -\frac{c\epsilon^2(1-\epsilon^2)[2(1-\xi^2)-(1-3\xi^2)\omega(\xi)]}{(1-\xi^2)^2+4\epsilon^2\xi^2+(1-\xi^2+2\epsilon^2\xi^2)\omega(\xi)}\,,\\ \sigma_{22}(\underset{\sim}{x}) &= -\frac{c\epsilon^2(1-\epsilon^2)[2(1-\xi^2)-(1+\xi^2)\omega(\xi)]}{(1-\xi^2)^2+4\epsilon^2\xi^2+(1-\xi^2+2\epsilon^2\xi^2)\omega(\xi)}\,.\end{aligned}\right\}\qquad(4.25)$$

Appendix: The associated punch problem.

We consider here briefly the problem of an elastic roller symmetrically indented by two flat and smooth rigid punches, as indicated in Figure 5. This problem is closely related to the contact problem treated in the body of the paper. In this connection we let ℓ now stand for the given punch width (Figure 5) and otherwise adhere to the notation introduced in Section 1. Accordingly, the symbols P, θ_o, θ_*, and Δ retain their previous meaning so that, in particular, $2\theta_o$ denotes the central angle subtended by each of the two circular arcs of ∂R that are brought into contact with the punch faces upon deformation of the roller.

It is clear to begin with that the punch problem and the original contact problem become identical if only a portion of either punch face ultimately comes into contact with the roller, i.e. if the applied load P is sufficiently small and ℓ suitably large. We therefore presuppose complete contact and eventually deduce an inequality that precludes the possibility of partial contact between either punch and the roller. Further, in the present circumstances we neglect the "vertical" displacements of the endpoints of each contact arc[1] and set

$$\ell=2r_o\sin\theta_o\,. \qquad (A1)$$

[1] Cf. equation (3.4), which is free from this approximation.

Within the preceding assumption the indentation problem at hand evidently obeys the formulation given in Section 1 of the contact problem considered there provided the separation condition (1.8) is relinquished and replaced by (A1), whence θ_o is at present known from the data.

On the basis of the reduction scheme adopted earlier it now follows that all of the equations in Section 1, with the exception of (1.8) and (1.25), continue to hold at present, the same being true of (2.1) through (2.18), while (2.19), (2.20), as well as any of their consequences, are no longer valid. Substitution from (1.49) into (1.26) yields

$$aE'+bK'=\frac{\pi k}{2r_o}P \ , \tag{A2}$$

which, together with the first of (2.16) and (A1), (1.20), gives

$$a=\frac{4r_o(2r_oK'+\pi kP)}{\ell^2K'+8r_o^2E'} \ , \quad b=\frac{\pi k\ell^2P-16r_o^3E'}{2r_o\left(\ell^2K'+8r_o^2E'\right)} \ . \tag{A3}$$

On the other hand, combining (A3) with the second of (2.16), one has

$$t_*=\frac{\pi}{2r_o}\,\frac{8r_o^3+kP\left[8r_o^2(E-K)-\ell^2K\right]}{\ell^2K'+8r_o^2E'} \ . \tag{A4}$$

Equations (A3), (A4) take the place of (2.22) in the previous problem.

From (A3), (1.50), (1.21), (A1) one obtains the pressure distribution appropriate to the indentation problem in the form

$$\left.\begin{aligned} &p(\theta)=\frac{8r_o^2(2r_oK'+\pi kP)t^2+\pi k\ell^2P-16r_o^3E'}{2\pi kr_o\left(\ell^2K'+8r_o^2E'\right)\sqrt{t^2-t_o^2}} \quad (0\le\theta<\theta_o) \ , \\ &t=\cos\theta \ , \quad t_o=\cos\theta_o=\sqrt{1-\ell^2/4r_o^2} \ , \quad k=\frac{1-\nu}{\pi\mu} \ , \end{aligned}\right\} \tag{A5}$$

in which K', E' are again the usual complete elliptic integrals for the complementary modulus $\sqrt{1-t_o^2} \equiv \ell/2r_o$. As is apparent, equation (A5) — in contrast to (3.1) — predicts that the contact pressure becomes unbounded as $\theta \to \theta_o$. The diametral compression for the punch problem is immediate from (A4) since, by (1.7) and (1.20),

$$\frac{\Delta}{2r_o} = 1 - t_* \ . \tag{A6}$$

Finally, according to (2.18), (1.20), (A1), and (A3), the contact pressure $p(\theta)$ is positive for $0 \le \theta < \theta_o$ if and only if

$$P \ge \frac{4r_o\left[4r_o^2 E' - \left(4r_o^2 - \ell^2\right)K'\right]}{\pi k\left(8r_o^2 - \ell^2\right)} \ , \tag{A7}$$

which is the desired necessary and sufficient condition for complete contact.

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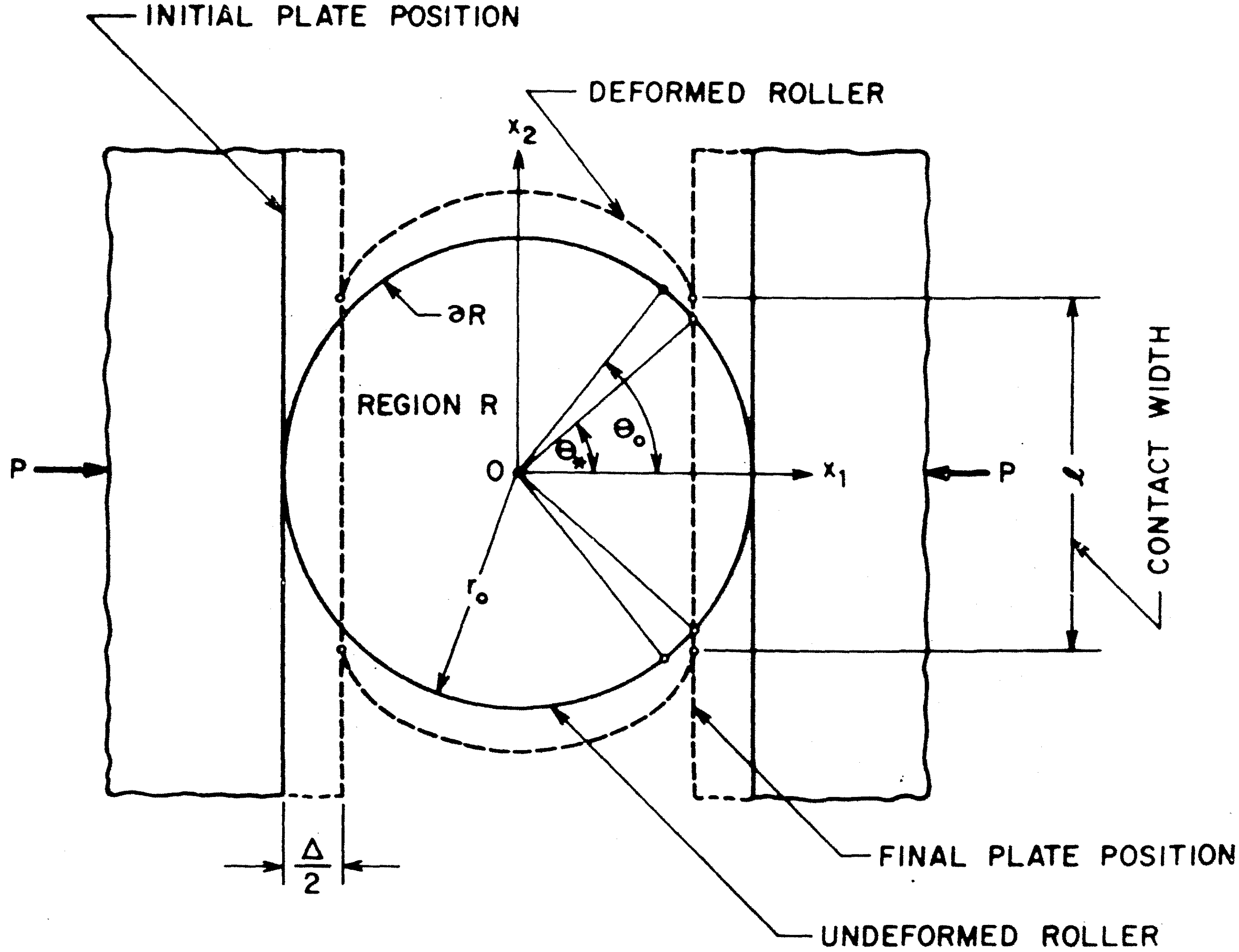

FIGURE 1. CIRCULAR ROLLER COMPRESSED BETWEEN RIGID PLATES.

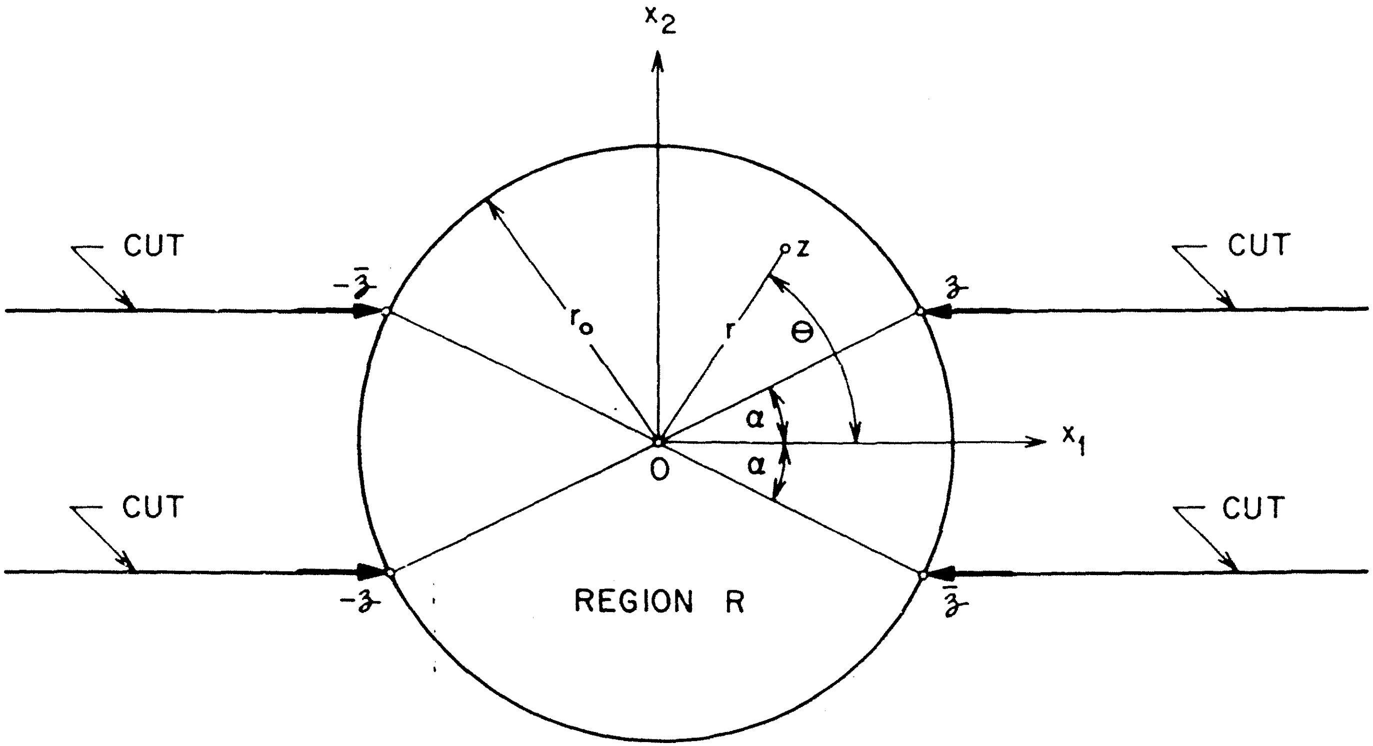

FIGURE 2. CROSS-SECTION IN THE COMPLEX z-PLANE OF ROLLER UNDER FOUR UNIT LINE-LOADS.

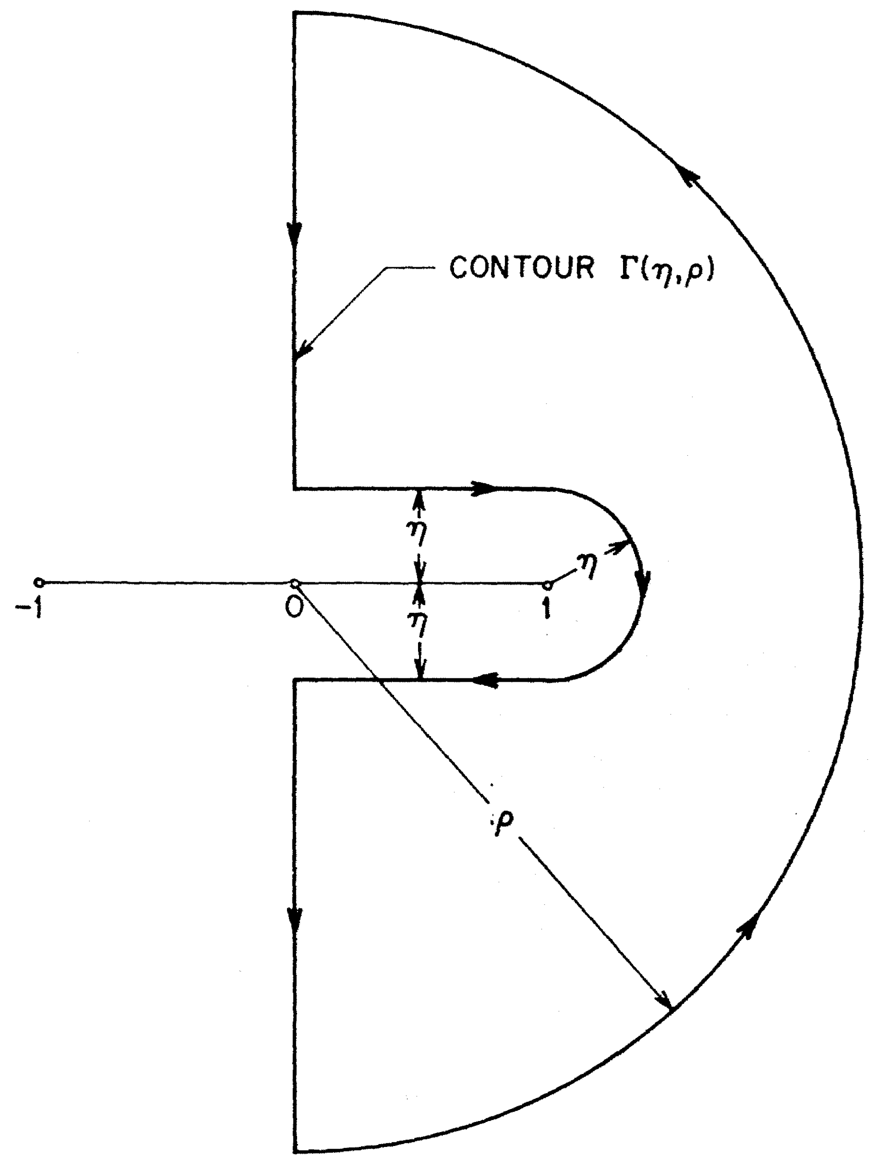

FIGURE 3. INTEGRATION PATH IN THE COMPLEX τ-PLANE FOR THE DETERMINATION OF w_1.

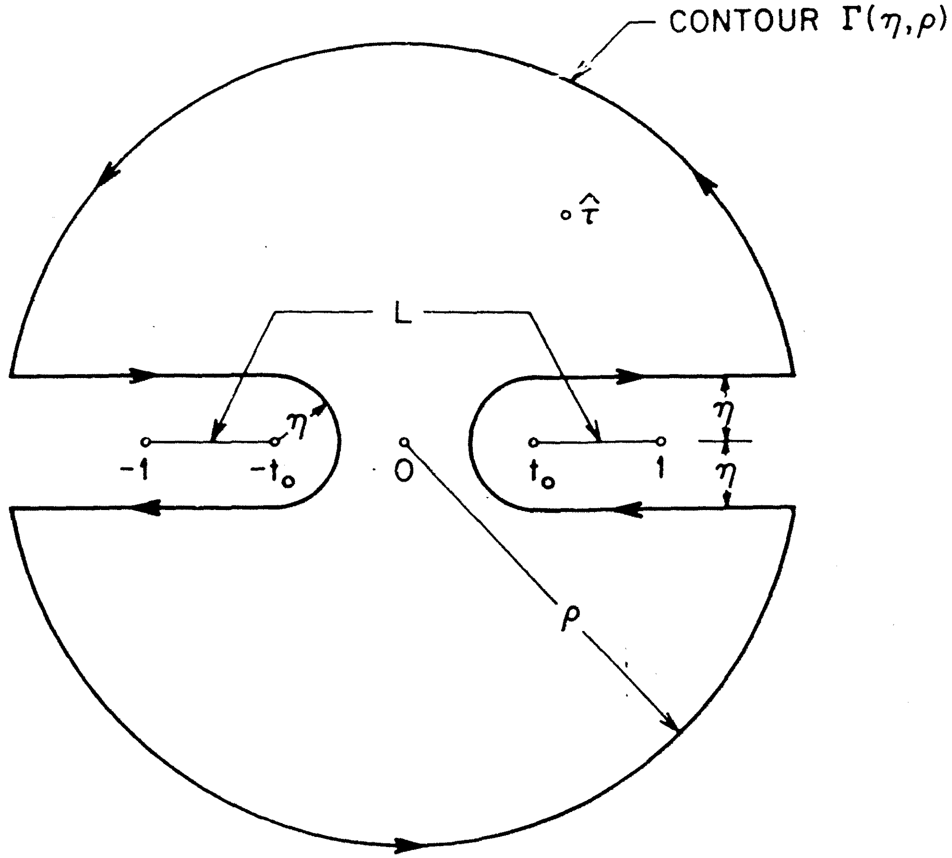

FIGURE 4. INTEGRATION PATH IN THE COMPLEX τ-PLANE FOR THE DETERMINATION OF THE STRESSES.

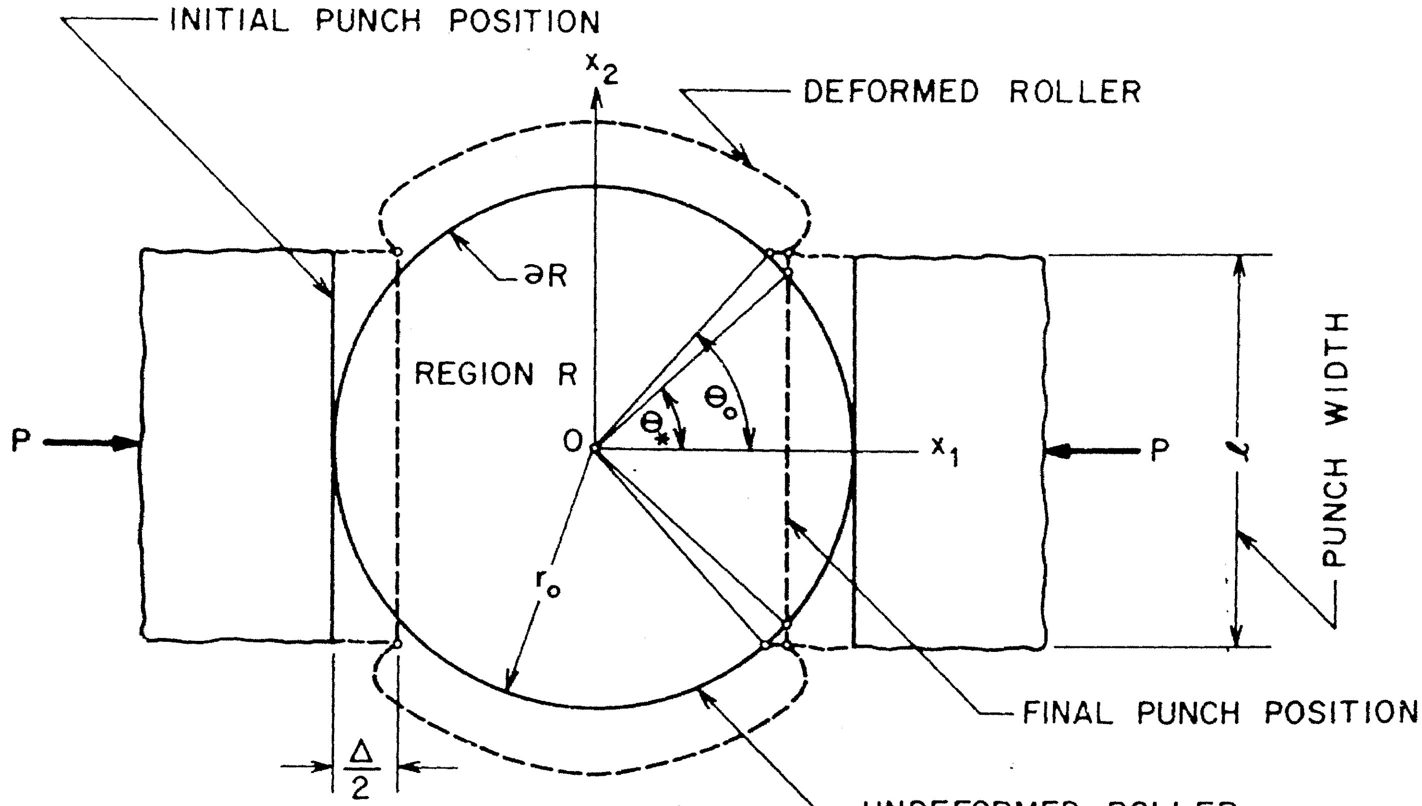

FIGURE 5. CIRCULAR ROLLER INDENTED BY RIGID PUNCHES.

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13. ABSTRACT

A closed solution — exact within two-dimensional linear elastostatics — is deduced for the problem appropriate to the compression of an elastic circular cylinder between two smooth, flat and parallel, rigid plates. The boundary displacements obtained for the cylinder involve elliptic integrals, whereas its stress field is given in terms of elementary functions exclusively. The results found for the distribution of the contact pressure and for the width of the contact zone are compared with the corresponding predictions of Hertz's approximate theory, for which elementary corrections are determined by asymptotic means. Analogous corrections are established for a previously available approximate estimate of the diametral compression undergone by the roller, which remains indeterminate in the Hertz treatment of this two-dimensional contact problem.

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